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$SU(2)_q$ in a Hilbert Space of Analytic Functions

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The algebra $SU(2)_q$ is realized in a Hilbert space H_q^2 of analytic functions; the starting point is the differential realization of operators that satisfy q-algebra in a Hilbert space H_q. The Weyl realization of $SU(2)_q$ is constructed exhibiting the reproducing kernel and the principal vectors; the noncommutativity of the matrix elements of a 2×2 linear representation of $SU(2)_q$ is obtained as consistency conditions for coupling $j1=j2=1/2$ to $j=0, 1$; the derivation of Clebsch-Gordan coefficients is sketched and the q-generalization of the rotation matrices is included. The unitary correspondence of H_q with a Hilbert space of complex functions of a real variable is also studied. The study presented in this paper follows Bargmann's formalism for the rotation group as closely as possible.

1. INTRODUCTION

1.1. The purpose of the present paper is to present a description of $SU(2)_q$ realized in a Hilbert space, denoted H_q^2, of analytic functions following (methodologically) as closely as possible Bargmann's scheme for the rotation group (Bargmann, 1962). The basic Hilbert space considered in the present work is a space, denoted H_q, of analytic functions of one complex variable on which a differential realization of the so-called q-algebra plays a central role; the differential realization of the q-algebra is the starting point for a Weyl representation of $SU(2)_q$ in the sense that $H_q^2 = H_q \otimes H_q$. The Hilbert space H_q is the same one proposed by Arik and Coon (1976) in connection with the construction of generalized coherent states. In this paper I add to the known mathematical structure of H_q explicit expressions for the reproducing kernel and the principal vectors. From the algebraic point of view the q-algebra is closely related to the one studied by Biedenharn (1989) and Macfarlane (1989) but is a wider scheme because

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in the last two reference $q \geq 0$, while the most general case allows $q \geq -1$ (Kuryshkin, 1980). In the present paper the parameter q is restricted to $0 < q < 1$.

1.2. The construction of the Hilbert space H_q^2 proceeds as follows: it is spanned by analytic functions in two complex variables with the inner product proposed by Arik and Coon (1976); as a by-product the Hilbert space is separated in orthogonal subspaces generated by homogeneous polynomials in the two complex variables. The bases of each of these invariant subspaces support the irreducible representations of $SU(2)_q$. The mathematical structure of the Hilbert space thus constructed proves to be similar to Bargmann's space for the rotation group: a reproducing kernel can be exhibited in an explicit way which reduces to the known result in the limit $q \to 1$; in addition, the characteristic vectors are explicitly constructed. The difference from Bargmann's space is that the functions belonging to H_q^2 are analytic in a finite region of C^2, where C represents the field of complex numbers. The next step [following Bargmann (1961)] is to relate H_q to a Hilbert space F of complex functions of a real variable (the equivalent of the Hilbert space of ordinary quantum mechanics). The q-algebra is realized in F if the inner product is not $\int f^*(x)g(x)\,dx$ but

$$f(d/dx)g(x)|x=0$$

The connection is then established with this Hilbert space F. It turns out that the transformation from H_q to F is unitary, as in Bargmann's study of the integral transform connecting the two Hilbert spaces considered by him (Bargmann, 1962); there is a small difference, however, which is that the unitarity is proved in the present case in a much easier way. It is not necessary to prove that the transform is isometric and then invertible to obtain unitarity; in the present case it is shown explicitly that a function in one space has one and only one image in the other, by arguing simply that both of them are characterized by the same set of parameters. The relation can be extended in an obvious way to connect H_q^2 with $F \otimes F$; for this reason this is not presented in this paper.

1.3. Next, the direct product of representations and the Clebsch expansion are studied. As compared with the rotation group, this study has a serious drawback: the general characteristic of the matrices of a representation of $SU(2)_q$ is unknown [in $SU(2)$ they are unitary] and therefore they cannot be used to define the way in which an element of $SU(2)_q$ changes an arbitrary analytic function. This was a crucial step in the study of the rotation group, which made it possible to obtain the matrix elements of all irreducible representations. Once they are known, the Clebsch-Gordan coefficients are easily computed. There are a number of ways to overcome this difficulty: one is to use invariant functions under the operators that satisfy $SU(2)_q$ and to expand these functions to compute the Clebsch-Gordan coefficients; another is the method of highest weight (used in this paper) to obtain the coefficients. Of course these coefficients have been computed in a number of papers and some of their properties studied in detail (Ruegg, 1990; Nomura, 1990*a*,*b*; Groza *et al.*, 1990). For this reason the development presented here goes up to the point where it makes contact with the current derivations. Subsequent calculations are not presented. Once the Clebsch-Gordan coefficients for $SU(2)_q$ are known, the noncommutativity of the matrix elements of the representation of $SU(2)_q$ is obtained.

1.4. As far as the mathematical apparatus is concerned, the so-called q-calculus (Jackson, 1951) and the comultiplication are essential ingredients for the computation of the tensor product of two representations. The q-calculus has as an important consequence the fact that the Leibniz rule has to be carefully applied. Finally, the result of all this exposition is that the study of $SU(2)_q$ is done in a unified way. No mention is made about the relations of $SU(2)_q$ to physical situations; its present study is tackled from the purely mathematical point of view.

1.5. Notation: $Di = D/Dzi$ and $\partial i = \partial/\partial zi$. In Section 3, the Clebsch-Gordan coefficients (CGC) C^{j1j2}_{m1m2m} and the q-matrices ${}_qD^{(j)}_{mm'}$ are written $C(m1, m2; j)$ and $D(m, m'; j)$, respectively; it is assumed that $j1$ and $j2$ are fixed. In the case $m = \pm 1/2$, $C(1/2, -1/2; j) = C(+-; j)$ and $D(1/2, -1/2; j) = D(+-; j)$, where $j = 0, 1$.

2. HILBERT SPACE FOR q-ALGEBRA (H_q)

Consider two operators A and $A^\dagger$, Hermitian conjugates of one another, that satisfy

$$AA^\dagger - qA^\dagger A = I \tag{2.1}$$

where q is a real parameter in the range $q \geq -1$. This is a family of algebras (one for each value of q and called q-algebras); they were studied by Kuryshkin (1980), who computed the matrix elements of A and $A^\dagger$ in a basis in which the operator $B \equiv A^\dagger A$ is diagonal. Before Kuryshkin's paper a study of (2.1) restricted to the range $0 < q < 1$ was reported by Arik and Coon (1976); their main purpose was to generalize the coherent states and realize the algebraic scheme in a Hilbert space of analytic functions. Further study on q-algebras is presented in Codriansky (1991), where the relation to para-Bose and para-Fermi algebras is clarified, a generalized Penney's theorem (Penney, 1965) is presented and a family of Hamiltonians exhibited.

2.1. Summary of Arik and Coon's (1976) Results

The parameter q is restricted to $0<q<1$; A and $A^\dagger$ are realized as differential operators acting on analytic functions $f(z)$ of a complex variable z as follows:

$$Af(z)=Df(z)/Dz\equiv\{f(z)-f(qz)\}/(1-q)z; \qquad A^\dagger f(z)=zf(z) \quad (2.2)$$

The "basic derivation" D/Dz was introduced by Jackson (1951) as the inverse of the "basic integration" defined as follows (x is a real variable):

$$\int_0^b F(x)\,Dx\equiv(1-q)b\sum_{l=0}^{\infty} q^l F(q^l b) \quad (2.3)$$

In terms of this basic integration the inner product in the space of analytic functions $f(z)$ and $g(z)$ of a complex variable z is defined as

$$(f,g)\equiv\pi^{-1}\int D^2z\, f^*(z)g(z)/\exp_q(q|z|^2) \quad (2.4)$$

where $z=|z|\,e^{i\phi}$ and f^* denotes the complex conjugate function

$$\int D^2z\, F(z,z^*)=(1/2)\int_0^{(\infty)} D(|z|^2)\int_0^{2\pi} d\phi\, F(z,z^*) \quad (2.5)$$

The q-exponential, $\exp_q(z)$, is defined as

$$\exp_q(z)=\sum_{n=0}^{\infty} z^n/[n]! \quad (2.6)$$

where

$$[n]\equiv(1-q^n)/(1-q), \qquad [\infty]=1/(1-q);$$
$$[n]!=[1]\bullet\cdots\bullet[n], \qquad [0]!=1 \quad (2.7)$$

(n a nonnegative integer) whose series expansion is seen to be convergent for $|z|^2<[\infty]$. This is the reason for the upper limit in (2.5). The space on which A and $A^\dagger$ act is the set of analytic functions $f(z)$ with convergent series expansions in the region $|z|^2<[\infty]$.

With the scalar product (2.4) the functions

$$u_m(z)=z^m/[n]! \quad (2.8)$$

are orthonormal and therefore form a basis of the space of functions. The space so constructed is a Hilbert space denoted H_q. The summary is ended.

Remark 2.1. In what follows I give a proof of the orthonormality of the functions $u_m(z)$ which is simpler than the one presented by Arik and Coon (1976). Compute the inner product (z^m, z^n),

$$(z^m, z^n) = \pi^{-1} \int D^2 z\, z^{*m} z^n / \exp_q(q|z|^2)$$

$$= \delta_{mn} \int_0^{(\infty)} D|z|^2\, |z|^{2m} / \exp_q(q|z|^2) \equiv \delta_{mn} I_m(q) \qquad (2.9)$$

now define (replace $|z|^2$ by u)

$$I_0(s) = \int_0^{[\infty]} Du/\exp_q(su) = -s^{-1} \qquad (2.10)$$

Then it follows that

$$D_s^m I_0(s) = (-1)^m q^{m(m-1)/2} \int_0^{[\infty]} Du\, u^m / \exp_q(sq^m u) \qquad (2.11)$$

so that

$$(-1)^m q^{-m(m-1)/2} D_s^m I_0(s)|_{s=q^{-m+1}} = I_m(q) \qquad (2.12)$$

Now, noting that $[-n] = -q^{-n}[n]$, it follows from (2.12) that $I_m(q) = [m]!$.

Remark 2.2. The space of functions so constructed is in fact a Hilbert space because if the function $f(z) = \sum_{n=0}^{\infty} a^n z^n$ is considered as being represented by the set of coefficients $\{a_n, n = 1, 2, \ldots\}$, then all sets that define convergent series expansions are included in the space; moreover, any set can be considered as the limit of a sequence of sets with a finite number of nonvanishing coefficients and therefore all limiting points are included. The space is complete and therefore a Hilbert space.

Remark 2.3. The inner product (2.4) can be derived in the same way as Bargmann (1961) obtained the inner product for the Hilbert space of entire functions that was subsequently used in the study of the rotation group. The main requirement is that the operators A and $A^\dagger$ be Hermitian conjugates under the inner product. Considering the differential realization (2.2), this requirement amounts to $(zf, g) = (f, Dg/Dz)$; at this point the inner product is defined as $(f, g) = K \int D^2 z\, \rho(z) f^*(z) g(z)$, where $\rho(z)$ is the kernel of the product and K is a constant. The equation that determines $\rho(z)$ is $D\rho(z/q)/Dz = -z^* \rho(z)$, which, after we write $\rho(z) = 1/R(q|z|^2)$, becomes $D(1/R(|z|^2))/Dz = -z^*/R(q|z|^2)$. The solution is (Jackson, 1951) $R(|z|^2) = \exp_q(|z|^2)$. In fact, from (2.7)

$$1/\exp_q(z) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} z^n / [n]! \qquad (2.13)$$

and from (2.2)

$$D(1/\exp_q(z))/Dz = -\sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2}(qz)^n/[n]! = -1/\exp_q(qz) \quad (2.14)$$

Remark 2.4. Basic differential operators other than (2.2) have been defined: for the particular cases $\alpha=-\beta=1/2$ (Ganchev and Petkova, 1989; Ruegg, 1990; Biedenharn, 1989; Gray and Nelson, 1990), and $\alpha=-\beta=1$ {Macfarlane (1989) defines only $[n]$, not the differential operator associated for this case; Sun and Fu (1990), Bracken *et al.* (1991). They are of the form (denoted generically by $\bar{D}$, α, and β real numbers)

$$\bar{D}f(z) \equiv (f(q^\alpha z)-f(q^\beta z))/(q^\alpha - q^\beta)z \quad (2.15)$$

Associated to each operator $\bar{D}$ is a particular "basic bracket" [the equivalent of (2.6)], n a nonnegative integer,

$$(n) \equiv (q^{\alpha n}-q^{\beta n})/(q^\alpha - q^\beta), \qquad (0)!=1, \qquad (n)! = (1)\bullet\cdots\bullet(n) \quad (2.16)$$

in terms of which a family of exponentials is defined, λ a parameter (real or complex), as

$$\exp_q^\lambda(z) = \sum_{n=0}^{\infty} q^{\lambda n(n-1)} z^n/(n)! \quad (2.17)$$

each of which satisfies $\bar{D}\exp_q^\lambda(z) = \exp_q^\lambda(q^{2\lambda}z)$.

Now, compute $\bar{D}(zf(z)) - h(q)z\bar{D}f(z)$, where $h(q)$ is an indetermined function of q that should be fixed so that $\bar{D}$ satisfies an algebraic relation as close as possible to a q-algebra. The result is $(d=d/dz)$

$$\bar{D}(zf(z)) - h(q)z\bar{D}f(z) = \{f(q^\alpha z)(q^\alpha - h(q)) - f(q^\beta z)(q^\beta - h(q))\}/(q^\alpha - q^\beta) \quad (2.18)$$

No function $h(q)$ allows recovering in the right-hand side the operator $\bar{D}$; on the other hand, the function $f(z)$ is recovered if $h(q)=q^\beta$ and $\alpha=0$. This is exactly a q-algebra with q replaced by q^β. If in (2.18) $f(q^\beta z)$ is replaced using (2.15), the result is

$$\bar{D}(zf(z)) - q^\beta z\bar{D}f(z) = f(q^\alpha z) = q^{\alpha zd}f(z) \quad (2.19)$$

which is independent of $h(q)$; therefore the extra freedom associated with the introduction of $h(q)$ is apparent. Now, (2.15), can be written as $\bar{D}f(z) = (q^\alpha - q^\beta)^{-1}(q^{\alpha zd} - q^{\beta zd})f(z)$, so that $[\![z\bar{D}, zd]\!]_- f(z)=0$. Define then the new operators $\tilde{D}f(z) \equiv (\bar{D}q^{-\alpha zd/2})f(z)$ and $\tilde{T}f(z) \equiv (q^{-\alpha zd/2}z)f(z)$. They satisfy the following commutation relation:

$$(\tilde{D}\tilde{T} - q^{\beta-\alpha/2}\tilde{T}\tilde{D})f(z) = f(z) \quad (2.20)$$

which is a q-algebra with $q \to q^{\beta-\alpha/2}$. As a result, all basic differentials defined in (2.15) satisfy a q-algebra whose parameter is determined by the particular values of α and β. The case considered in equation (2.2) is the simplest in the sense that a q-algebra is obtained without redefinition of the operators. In any case, all values of α and β such that $\alpha-\beta/2=1$ define the same algebra as (2.2).

2.2. The Reproducing Kernel and the Principal Vectors in H_q

The reproducing kernel (Aronszajn, 1950) is defined as a function $K(t, z)$ of two complex variables (t, z), which for every z belongs to the Hilbert space H_q and for every function $f \in H_q$

$$f(z)=(K(t, z), f(t))_t \tag{2.21}$$

where the subscript in the scalar product indicates that it applies to functions of t (Aronszajn, 1950). Since the monomials (2.8) are orthonormal, it follows that

$$K(t, z)=\sum_{m=0}^{\infty} t^m z^{*m}/[m]! = \exp_q(tz^*) \tag{2.22}$$

is the reproducing kernel in H_q.

The principal vector e_α is defined as follows (α a complex number):

$$f(\alpha)=(e_\alpha, f) \tag{2.23}$$

Its explicit form as a function of α and z $(\alpha, z \in C)$ is

$$e_\alpha(z)=\sum_{k=0}^{\infty} \alpha^{*k} z^k/[k]! = \exp_q(\alpha^* z) \tag{2.24}$$

Expressions (2.22) and (2.24) reduce to Bargmann's result (Bargmann, 1961) when $q \to 1$. The principal vectors belong to H_q^*, the dual of H_q.

Remark 2.5. A construction of a Hilbert space similar to the one sketched in this section was reported by Bracken *et al.* (1991). Their expression for the inner product is, however, different mainly because they define it with the q-exponential used in their paper as $E(-r^2)$ (their notation) in the numerator. In this way the behavior of the exponential for negative values of the argument governs the inner product; in the inner product I studied, positive values of the argument come into play. An important difference with the present approach is that Bracken *et al.* (1991) start by generalizing directly the known results for coherent states and then construct the underlying algebra. I start with the q-algebra and as a consequence all developments are simpler, in particular (and this is one of the main points) the derivation of the kernel of the inner product.

3. HILBERT SPACE FOR $SU(2)_q$

3.1. The Weyl Realization

Consider the linear space spanned by the function [Ruegg (1990) and Sun and Fu (1990) considered these same functions]

$$v_m^j(z_1, z_2) = M(j, m) z_1^{j+m} z_2^{j-m} \tag{3.1}$$

where $M(j, m) = \{[j+m]!\,[j-m]!\}^{-1/2}$ and (z_1, z_2) are a pair of complex variables restricted by $|z_i|^2 < [\infty]$, $i = 1, 2$. Define the scalar product in this space by

$$(f, g) = \pi^{-2} \int Dz_1\, Dz_2 \frac{f^*(z_1, z_2) g(z_1, z_2)}{\exp_q(q|z_1|^2) \exp_q(q|z_2|^2)} \tag{3.2}$$

Then, it is clear that the monomials (3.1) are orthonormal under (3.2). The linear space consists of functions that can be expanded in a Taylor series convergent in the region $\{(z_1, z_2), |z_i|^2 < [\infty], i = 1, 2\}$. Completeness is proved in the usual way; therefore, the linear space with the scalar product (3.2) is a Hilbert space H_q^2, which is the tensor product of two copies of H_q, $H_q^2 = H_q \otimes H_q$. It is in this space that the Weyl realization of $SU(2)_q$ will be exhibited.

The differential operators $J\pm$, Jz, and C defined as

$$J+ = q^{(1-C)/4} z_1 D_2, \qquad J- = q^{(1-C)/4} z_2 D_1 \tag{3.3a}$$

$$Jz = (z_1\partial_1 - z_2\partial_2)/2, \qquad C = z_1\partial_1 + z_2\partial_2 \tag{3.3b}$$

yield, when acting on $v_m^j(z_1, z_2)$,

$$\begin{aligned}(J+) v_m^j(z_1, z_2) &= q^{(1-2j)/4}[j-m] M(jm) z_1^{j+m+1} z_2^{j-m-1} \\ &= q^{(1-2j)/4}\{[j-m][j+m+1]\}^{1/2} \\ &\quad \times v_{m+1}^j(z_1, z_2)\end{aligned} \tag{3.4a}$$

$$\begin{aligned}(J-) v_m^j(z_1, z_2) &= q^{(1-2j)/4}\{[j+m][j-m+1]\}^{1/2} \\ &\quad \times v_{m-1}^j(z_1, z_2)\end{aligned} \tag{3.4b}$$

$$(Jz) v_m^j(z_1, z_2) = m v_m^j(z_1, z_2) \tag{3.4c}$$

$$C v_m^j(z_1, z_2) = 2j v_m^j(z_1, z_2) \tag{3.4d}$$

and from here ($[\![a, b]\!]_- \equiv ab - ba$)

$$[\![J\pm, Jz]\!]_- v_m^j(z_1, z_2) = \pm(J\pm) v_m^j(z_1, z_2) \tag{3.5a}$$

$$\begin{aligned}[\![J+, J-]\!]_- v_m^j(z_1, z_2) &= (q^{Jz} - q^{-Jz})/(q^{1/2} - q^{-1/2}) \\ &\quad \times v_m^j(z_1, z_2)\end{aligned} \tag{3.5b}$$

$$[\![C, J\pm]\!]_- = 0, \qquad [\![C, Jz]\!]_- = 0 \tag{3.6}$$

Relations (3.5a) and (3.5b) are the $SU(2)_q$ commutation relations; (3.5b) is obtained when acting on $v_m^j(z_1, z_2)$, not generically (Biedenharn, 1989), while (3.5a) is generically obtained. C is a Casimir operator as follows from (3.6); it is the $SU(2)_q$ generalization of the so-called Euler operator of $SU(2)$ (Biedenharn and Louck, 1981). It follows that the linear subspace $H_q^2(j)$ of homogeneous polynomials of degree $2j$ is invariant under $J\pm$, Jz; moreover, this subspace is orthogonal to any linear subspace of homogeneous polynomials of degree $2j'$ $(j \neq j')$. In this way it is found that H_q^2 is a separable Hilbert space; $H_q^2 = \oplus H_q^2(j)$.

Remark 3.1. The operators $J\pm$ defined in (3.3) differ in the factor $q^{(1-C)/4}$ from the ones usually defined (see, for instance, Ruegg, 1990). The reason lies in the fact that I am constructing them as functions of operators satisfying the q-algebra (2.1) realized as in (2.2) with the bracket (2.6).

Remark 3.2. If instead of the Casimir operator C defined in (3.3) we consider

$$C' = (z_1\partial_1 + z_2\partial_2)(z_1\partial_1 + z_2\partial_2 + 2)/4 \tag{3.7}$$

then its eigenvalues are $j(j+1)$ when acting on $v_m^j(z_1, z_2)$. On taking the limit $q \to 1$, C' remains the same for all values of q and only $J\pm$ are deformed into the corresponding $SU(2)$ operators.

3.2. The Reproducing Kernel and Principal Vectors in H_q^2

The explicit expression for the reproducing kernel in H_q^2 is

$$\begin{aligned} K(s_1, s_2; z_1, z_2) &= \sum_{2j=0}^{\infty} \sum_{m=-j}^{j} v_m^j{}^*(s_1, s_2) v_m^j(z_1, z_2) \\ &= \exp_q(s_1^* z_1) \exp_q(s_2^* z_2) \end{aligned} \tag{3.8}$$

and for the principal vectors $e_{\alpha 1, \alpha 2}$ is

$$e_{\alpha 1, \alpha 2}(z_1, z_2) = \exp_q(\alpha 1^* z_1) \exp_q(\alpha 2^* z_2) \tag{3.9}$$

It is easy to verify that $K(s_1, s_2; z_1, z_2)$ belongs to H_q^2 for each pair (s_1, s_2); in fact, the reproducing property implies

$$\begin{aligned} &K(s_1, s_2; z_1, z_2) \\ &\quad = (K(s_1, s_2; \alpha_1, \alpha_2), K(\alpha_1, \alpha_2; z_1, z_2))_{\alpha 1, \alpha 2} \end{aligned} \tag{3.10}$$

so that for $z_1 = s_1$ and $z_2 = s_2$

$$\begin{aligned} \|K(s_1, s_2; s_1, s_2)\|^2 &= K(s_1, s_2; s_1, s_2) \\ &= \exp_q(|s_1|^2) \exp_q(|s_2|^2) < \infty \end{aligned} \tag{3.11}$$

From (3.8) it follows that the restriction of the reproducing kernel to the closed linear subspace corresponding to a fixed value of j is

$$K^{(j)}(s_1, s_2; z_1, z_2) = \sum_{m=-j}^{j} v_m^j{}^*(s_1, s_2) v_m^j(z_1, z_2) \tag{3.12}$$

at the same time $K^{(j)}(s_1, s_2; z_1, z_2)$ is a projection operator onto the subspace of homogeneous polynomials of order $2j$.

3.3. A Linear Representation of $SU(2)_q$

To every element $g \in SU(2)_q$ is associated a 2×2 matrix U with coefficients whose properties have to be determined. As in Bargmann (1962), a representation is obtained if for every $f \in H_q^2$

$$(Uf)(\psi) = f(U^t \psi) \tag{3.13}$$

where the superscript in U^t denotes the transpose and ψ is the column vector with $\psi_1 = z_1$, $\psi_2 = z_2$. If the matrix elements of U are denoted U_{il}, $i, l = 1, 2$ (i = row, l = column), then for $f(\psi) = v_m^j(\psi)$

$$(Uv_m^j)(\psi) = M(j, m)(U_{11}z_1 + U_{21}z_2)^{j+m}(U_{12}z_1 + U_{22}z_2)^{j-m} \tag{3.14}$$

and from this the matrix elements of the representation $D(m, m'; j)(U)$ are

$$\begin{aligned} &D(m, m'; j)(U) \\ &= (v_{m'}^j, Uv_m^j) \\ &= M(j, m')M(j, m) \sum_{t=0}^{j-m'} \sum_{k=0}^{j+m'} \frac{[j+m']!\,[j-m']!}{[k]!\,[j+m'-k]!\,[t]!\,[j-m']!} \\ &\times D_2^{j-m'-t} D_1^{j+m'-k}(U_{11}q^k z_1 + U_{21}q^t z_2)^{j+m}\big|_{\psi=0} \\ &\times D_2^t D_1^k (U_{12}z_1 + U_{22}z_2)^{j-m}\big|_{\psi=0} \end{aligned} \tag{3.15}$$

An easy way to prove (3.15) is to use the fact that the inner product (2.4) can be computed using a differential instead of the integral form of the definition. In fact, the monomials (2.8) are orthonormal under the inner product

$$(u_m, u_n) \equiv u_m(D/Dz)u_n(z)\big|_{z=0} \tag{3.16}$$

so that the same numerical result is obtained using both versions of the inner product [(2.4) and (3.16)]. Computing with (3.16) and using the q-version of the Leibnitz rule,

$$\frac{D}{Dz}\{f(z)g(z)\} = \{Df(z)/Dz\}g(z) + f(qz)\frac{Dg(z)}{Dz} \tag{3.17a}$$

$$\frac{D^n}{Dz^n\{f(z)g(z)\}} = \sum_{k=0}^{n} \frac{[n]!}{[k]!\,[n-k]!} \frac{D^k f(q^k z)}{Dz^k} \frac{D^{n-k}g(z)}{Dz^{n-k}} \tag{3.17b}$$

we easily obtain the result (3.15). It is proved in Section 3.5 that the coefficients U_{il} are not ordinary numbers, because they do not commute; this result is obtained as a consistency condition for the coupling of two copies of $SU(2)_q$ [of course this property of the coefficients is part of the current knowledge of $SU(2)_q$].

3.4. The Space H_q^4

This space is spanned by functions of four complex variables (z_i, $i = 1, \ldots, 4$) in the region defined by $|z_i|^2 < [\infty]$. A basis in this space is $v_{m1}^{j1}(\psi_1) v_{m2}^{j2}(\psi_2)$, where the ψ_i are the vectors defined in the previous section. The operators that satisfy the $SU(2)_q$ commutation relations are (Ruegg, 1990)

$$Jz = J1z \otimes I_2 + I_1 \otimes J2z; \qquad J\pm = J1\pm \otimes q^{J2z/2} + q^{J1z/2} \otimes J2\pm \quad (3.18)$$

where (Jiz, $Ji\pm$) operate on $v_{mi}^{ji}(\psi_i)$, $i = 1, 2$, as indicated in equations (3.4a)–(3.4c) and I_i is the identity in the ith space. Another basis in H_q^4 is the set of functions $f_m^j(\psi 1, \psi 2)$ such that Jz and $J\pm$ act on these as in equations (3.4a)–(3.4c); these functions can, of course, be expanded in terms of $v_{m1}^{j1}(\psi_1) v_{m2}^{j2}(\psi_2)$. Both sets of functions are related through the Clebsch–Gordan coefficients (CGC). Since explicit expressions for the CGC have been published (Groza *et al.*, 1990, and references therein), they will not be repeated here; only some of the initial and final steps in the derivation are exhibited below to point out where minor differences arise.

For the derivation I use the method of highest weight, which starts by expanding $f_j^j(\psi 1, \psi 2)$ as

$$f_j^j(\psi 1, \psi 2) = \sum_{m1=-j1}^{j1} \sum_{m2=-j2}^{j2} A_{m1,m2} v_{m1}^{j1}(\psi_1) v_{m2}^{j2}(\psi_2) \qquad (3.19)$$

where $j1, j2$ are fixed, $|j1 - j2| \leq j \leq j1 + j2$, and the $A_{m1,m2}$ are coefficients determined by the condition $J + f_j^j(\psi 1, \psi 2) = 0$; this gives a recurrence relation whose final result is (Biedenharn and Louck, 1981)

$$A_{m1,m2} = (-1)^{j1-m1} q^{-(j1-m1)(m1+m2+1)/2} q^{-(j1-m1)(j2-j1)/2} \times \{[j1+m1]!\,[j2+m2]!/[j1-m1]!\,[j2-m2]!\}^{1/2} A \qquad (3.20)$$

where A is a constant fixed by the condition $(f_j^j, f_j^j) = 1$. Equation (3.20) differs from the result of Groza *et al.* (1990) in the factor $q^{-(j1-m1)(j2-j1)/2}$, which arises in the present case as a consequence of the definition of the operators $J\pm$ [see equation (3.3)]. This difference modifies also the constant A to

$$A = \left\{ \sum_{\substack{m1,m2 \\ m1+m2=j}} q^{-(j1-m1)(j+1)} q^{-(j1-m1)(j1-j2)} \frac{[j1+m1]!\,[j2+m2]!}{[j1-m1]!\,[j2-m2]!} \right\}^{-1/2} \qquad (3.21)$$

Remark 3.3. The closed expression of Groza *et al.* (1990) for A [which appears before their equation (39)] seems to be wrong. In fact, if $l1 = l2 = 1/2$ and $l = 0$, the results are $A^{-2} = 1 + q^{-1/2}$ [from their formula after equation (38)] and $A^{-2} = q[2] = q(q^{1/2} + q^{-1/2})$ [from their formula before equation (39)], which are obviously different. Of course, to compute [2] I used their expression $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2}q^{-1/2})$ for the basic bracket. Their result after equation (38) is correct and leads to the known value when $q \to 1$. In any case, I cannot present a closed expression for A.

The other members of the multiplet $(f^j_m(\psi 1, \psi 2))$ are given as

$$f^j_m(\psi 1, \psi 2) = q^{-(1-2j)(j-m)/4}\{[j+m]!/[2j]!\,[j-m]!\}^{1/2} \times (J-)^{j-m} f^j_j(\psi 1, \psi 2) \qquad (3.22)$$

where the factor $q^{-(1-2j)(j-m)/4}$ comes from the definition of $J\pm$ [equation (3.3)]. The CGC are

$$\begin{aligned} C(m1, m2; j) &= (v^{j1}_{m1} v^{j2}_{m2}, f^j_m) \\ &= q^{-(1-2j)(j-m)/4}\{[j+m]!/[2j]!\,[j-m]!\}^{1/2} \\ &\quad \times \sum_{\substack{r,p \\ r+p=j}} A_{rp}((J+)^{j-m} v^{j1}_{m1} v^{j2}_{m2},\, v^{j1}_r v^{j2}_p) \end{aligned} \qquad (3.23)$$

whose explicit expression is

$$\begin{aligned} C(m1, m2; j) &= q^{-(j-j2)(j-m)/2}\{[j+m]!/[2j]!\,[j-m]!\}^{1/2} \\ &\quad \times \sum_{k=0}^{j-m} F(j-m, k; j1, m1; j2, m2) A_{m1+k, m2+j-m-k} \end{aligned} \qquad (3.24)$$

where

$$\begin{aligned} &F(j-m, k; j1, m1; j2, m2) \\ &= q^{k(j+j2-j1-k)/2}[j-m]!/[k]!\,[j-m-k]! \\ &\quad \times \{[j1-m1]!\,[j1+m1+k]!\,[j2-m2]!\,[j2+m2+j-m-k]!\}^{1/2} \\ &\quad \times \{[j1+m1]!\,[j1-m1-k]!\,[j2+m2]!\,[j2-m2-j+m-k]!\}^{-1/2} \end{aligned} \qquad (3.24a)$$

3.5. The Representation $D(m, m'; 1/2)$

The representation matrices $D(m, m'; j)(U)$ were introduced in Section 3.3. In this section the conditions satisfied by the matrix elements of the representation with $j=1/2$ are obtained. The starting point is the construction of an invariant, namely, a function $h(\psi 1, \psi 2)$ such that $(Uh)(\psi 1, \psi 2) = h(\psi 1, \psi 2)$; the explicit expression for $h(\psi 1, \psi 2)$ is

$$h(\psi 1, \psi 2) = \sum_{m=-j}^{j} C(m, -m; 0) v_m^j(\psi 1) v_{-m}^j(\psi 2) \tag{3.25}$$

and from (3.12)

$$(Uh)(\psi 1, \psi 2) = \sum_{m,m',m''=-j}^{j} C(m, -m; 0) D(m, m'; j) D(-m, m'', j) \times v_{m'}^j(\psi 1) v_{m''}^j(\psi 2) \tag{3.26}$$

Equating (3.25) and (3.26) gives

$$\sum_{m=-j}^{j} C(m, -m; 0) D(m, m1; j) D(-m, m2; j) = C(m1, m2; 0)\delta_{m1,-m2} \tag{3.27}$$

which is a relation involving the $D(m, m'; j)$; it will be used to obtain the conditions satisfied by the matrix elements of the fundamental representation $D(m, m'; 1/2)$.

Incidentally, (3.25) can be used to obtain a recurrence relations for the CGC; this is done by noting that $J+h=0$. Thus, for $j=1/2$ it implies

$$C(+-; 0) = -q^{1/2} C(-+; 0) \tag{3.28}$$

If in (3.27) $j=1/2$, then four equations are obtained which, after use of (3.28), reduce to

$$D(-+; 1/2) D(++; 1/2) = q^{1/2} D(++; 1/2) D(-+; 1/2) \tag{3.29a}$$

$$D(--; 1/2) D(+-; 1/2) = q^{1/2} D(+-; 1/2) D(--; 1/2) \tag{3.29b}$$

$$D(--; 1/2) D(++; 1/2) - q^{1/2} D(+-; 1/2) D(-+; 1/2) = 1 \tag{3.29c}$$

$$D(-+; 1/2) D(+-; 1/2) - q^{1/2} D(++; 1/2) D(--; 1/2) = -q^{1/2} \tag{3.29d}$$

Notice that (3.29c) is the so-called quantum determinant condition.

Now couple $j1 = j2 = 1/2$ to $j=1$; then

$$v_1^1(\psi_1, \psi_2) = v_{1/2}^{1/2}(\psi_1) v_{1/2}^{1/2}(\psi_2) \tag{3.30a}$$

$$v_0^1(\psi_1, \psi_2) = C(+-; 1) v_{1/2}^{1/2}(\psi_1) v_{-1/2}^{1/2}(\psi_2) + C(-+; 1) v_{1/2}^{1/2}(\psi_1) v_{-1/2}^{1/2}(\psi_2) \tag{3.30b}$$

$$v_{-1}^1(\psi_1, \psi_2) = v_{-1/2}^{1/2}(\psi_1) v_{-1/2}^{1/2}(\psi_2) \tag{3.30c}$$

If $J+$ acts on $v_0^1(\psi_1, \psi_2)$ the result is

$$(J+)v_0^1(\psi_1, \psi_2) = q^{-1/4}[2]^{1/2}v_1^1(\psi_1, \psi_2)$$
$$= \{q^{-1/4}C(+-; 1) + q^{1/4}C(-+; 1)\}$$
$$\times v_{1/2}^{1/2}(\psi_1)v_{1/2}^{1/2}(\psi_2) \quad (3.31)$$

which implies

$$q^{-1/4}C(+-; 1) + q^{1/4}C(-+; 1) = q^{-1/4}[2]^{1/2} \quad (3.32)$$

The normalization of $v_0^1(\psi_1, \psi_2)$ gives a second equation to determine the two CGC involved; the result is

$$C(+-; 1) = (1+q)^{-1/2}; \qquad C(-+; 1) = q^{1/2}(1+q)^{-1/2} \quad (3.33)$$

Acting on $v_1^1(\psi_1, \psi_2)$ with an element $U \in SU(2)q$ gives the following relation:

$$\sum_{m=-1}^{1} D(1, m; 1)v_m^1(\psi_1, \psi_2)$$
$$= \sum_{m=-1}^{1} D(1, m; 1) \sum_{m'=-1/2}^{1/2} C(m', m-m'; 1)v_{m'}^{1/2}(\psi_1)v_{m-m'}^{1/2}(\psi_2)$$
$$\times \sum_{m', m''=-1/2}^{1/2} D(+, m'; 1/2)D(+, m''; 1/2)v_{m'}^{1/2}(\psi_1)v_{m''}^{1/2}(\psi_2) \quad (3.34)$$

from which the following four equations are obtained:

$$D(1, -1; 1) = D(+-; 1/2)D(+-; 1/2) \quad (3.35a)$$

$$\mathrm{D}(1, 1; 1) = \mathrm{D}(++; 1/2)\mathrm{D}(++; 1/2) \quad (3.35b)$$

$$D(1, 0; 1)C(-+; 1) = D(+-; 1/2)D(++; 1/2) \quad (3.35c)$$

$$D(1, 0; 1)C(+-; 1) = D(++; 1/2)D(+-; 1/2) \quad (3.35d)$$

From (3.35c) and (3.35d) it follows that [using (3.33)]

$$D(++; 1/2)D(+-; 1/2) = q^{-1/2}D(+-; 1/2)D(++; 1/2) \quad (3.36)$$

This relation ensures that equations (3.35c) and (3.35d) are simultaneously satisfied; it is in this sense a consistency condition. Proceeding in a similar way with $v_0^1(\psi_1, \psi_2)$, two equations involving $D(0, 0; 1)$ are obtained; from these it follows that

$$D(+-; 1/2)D(-+; 1/2) = D(-+; 1/2)D(+-; 1/2) \quad (3.37)$$

and finally, from $v_{-1}^1(\psi_1, \psi_2)$,

$$D(--; 1/2)D(-+; 1/2) = q^{1/2}D(-+; 1/2)D(--; 1/2) \quad (3.38)$$

Summarizing: Equations (3.29a)-(3.29d) and (3.36)-(3.38) are the seven basic relations satisfied by the matrix elements of $D(m, m'; 1/2)(U)$.

Since for the fundamental representation ${}_qD(1/2)(U) = U$ and the conditions satisfied by the matrix elements are now known, they can be used in (3.15) to compute $D(m, m'; j)(U)$. The result is

$$\begin{aligned} &D(m, m'; j)(U) \\ &= M(j, m')M(j, m) \sum_{t=0}^{j-m'} \sum_{k=0}^{j+m'} \sum_{r=0}^{j+m} \sum_{p=0}^{j-m} \\ &\times \frac{[j+m']!\,[j-m']!\,[j+m]!\,[j-m]!}{[k]!\,[j+m'-k]!\,[t]!\,[j-m'-t]!\,[r]!\,[p]!\,[j+m-r]!\,[j-m-p]!} \\ &\times q^{k(j+m-r)} U_{11}^{j+m-r} U_{21}^{r} U_{12}^{j-m-p} U_{22}^{p} \\ &\times \prod_{r'=0}^{j+m'-k-1} [j+m-r-r'] \prod_{r''=0}^{j-m'-t-1} [r-r''] z_1^{m-m'-r+k} z_2^{r-j+m'+t}\Big|_{z1=z2=0} \\ &\times \prod_{s'=0}^{k-1} [j-m-p-s'] \prod_{s''=0}^{t-1} [p-s''] z_1^{j-m-p-k} z_2^{p-t}\Big|_{z1=z2=0} \end{aligned} \qquad (3.39)$$

4. A HILBERT SPACE RELATED TO H_q

In this section, I tackle the following question: Is there a Hilbert space F of complex functions of a real variable x unitarily related to H_q? The space F plays the role of the Hilbert space of ordinary quantum mechanics. To establish the correspondence, it is necessary to define the inner product of F in such a way that for a pair of functions $\phi(x)$, $\psi(x) \in F$ and a pair of operators O, $O^\dagger$

$$(O\phi, \psi)_F = (\phi, O^\dagger\psi)_F \qquad (4.1)$$

where $(\cdot, \cdot)_F$ is the inner product in F and $OO^\dagger - qO^\dagger O = I$. The first step is, therefore, to determine O and $O^\dagger$ that satisfy the q-algebra, and next to establish the way in which a function in H_q is related to a function in F and finally to require that

$$(Df/Dz, g) = (O\phi, \psi)_F = (f, zg) = (\phi, O^\dagger\psi)_F \qquad (4.2)$$

The operators O, $O^\dagger$ act on functions in F as follows:

$$O\phi(x) \equiv D\phi(x)/Dx, \qquad O^\dagger\phi(x) \equiv x\phi(x) \qquad (4.3)$$

and the inner product that satisfies (4.1) is

$$(\phi, \psi)_F \equiv \phi(D/Dx)\psi(x)|x=0 \qquad (4.4)$$

Remark 4.1. It is easy to check that no inner product of the form $(\phi, \psi)'_F = \int_{-\infty}^{\infty} \rho(x)\phi^*(x)\psi(x)\, Dx$ satisfying (4.1) and the q-algebra exists; in fact, from (4.1)

$$(D\phi/Dx, \psi)'_F = \int_{-\infty}^{\infty} \rho(x)\, D\phi^*(x)/Dx\, \psi(x)\, Dx$$
$$= (\phi, x\psi)'_F = \int_{-\infty}^{\infty} \rho(x)\phi^*(x)x\psi(x)\, Dx \tag{4.5}$$

so that the equation satisfied by $\rho(x)$ is [recall the q-Leibnitz rule (3.17a), (3.17b)]

$$\int_{-\infty}^{\infty} \phi^*(x)\{x\rho(x) + D\rho(x/q)/Dx\}\psi(x)\, Dx$$
$$= \int_{-\infty}^{\infty} \phi^*(qx)\rho(x)\, D\psi(x)/Dx\, Dx \tag{4.6}$$

and to obtain (4.6) it has been assumed that $\phi^*(x)\rho(x)\psi(x) \to 0$ as $|x| \to \infty$. The equation that determines $\rho(x)$ should not depend on the functions $\phi(x)$ and/or $\psi(x)$ and this condition cannot be satisfied in (4.6).

The monomials $x^m/([m]!)^{1/2}$ are orthonormal under the inner product (4.4); therefore the Hilbert space F is the set of all functions $\psi(x)$ which can be expanded in a Taylor series

$$\psi(x) = \sum_{m=0}^{\infty} \alpha_m x^m/([m]!)^{1/2} \tag{4.7}$$

such that $\sum_{m=0}^{\infty} |\alpha|^2 < \infty$. The inner product of $\phi(x)$ and $\psi(x)$ is then

$$(\psi, \phi)_F = \sum_{m=0}^{\infty} \alpha_m^* \beta_m \tag{4.8}$$

With this result it is an easy matter to construct the operator that relates a function $\psi(x) \in F$ to a function $f(z) \in H_q$. Denoting it as $A(z, x)$, its expression is

$$A(z, x) = \sum_{m=0}^{\infty} z^{*m} x^m/[m]! \tag{4.9}$$

and the relation is as follows:

$$f(z) = (A(z, x), \psi(x))_x, \qquad \psi(x) = (A(z^*, x), f(z))_{F,z} \tag{4.10}$$

As a result, both functions $f(z) \in H_q$ and $\psi(x) \in F$ are characterized by the same set of coefficients in the series expansions and therefore the correspondence is one-to-one. It also follows that under the correspondence $A(z, x)$: $\psi \to f$

$$(f, f) = (\psi, \psi)_F \tag{4.11}$$

The final result of the above discussion is that both Hilbert spaces (F and H_q) are unitarily related.

To close this section, I prove that under the inner product (4.4) the operators introduced in (4.3) are Hermitian conjugates of one another. In fact, consider two functions $\psi(x) = \sum \alpha_m x^m/([m]!)^{1/2}$ and $\phi(x) = \sum \beta_n x^n/([n]!)^{1/2}$; then

$$(D\psi/Dx, \phi)_F = \sum_{n=1}^{\infty} [n+1]!\, \alpha_{n+1}^* \beta_n = (\psi, x\phi)_F \tag{4.12}$$

which proves the assertion.

5. SUMMARY AND RESULTS

In this paper I have constructed a Hilbert space (H_q^2) of analytic functions that resembles as closely as possible Bargmann's (1961, 1962) construction for the rotation group. The CGC have been computed and some minor differences with existing results pointed out; all these differences are due to the way in which the space H_q was defined; as a by-product of the study of H_q^4 the noncommutative properties of the matrix elements of a linear representation of $SU(2)_q$ were obtained as consistency conditions when coupling $j1 = j2 = 1/2$ to $j = 0, 1$; with this result at hand, explicit expressions for the matrices of the irreducible representation of order $(2j+1)$ were exhibited. As a final result, the relation of H_q to the Hilbert space F of complex functions of a real variable was established; the inner product in F is not the product in the ordinary Hilbert space of quantum mechanics.

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